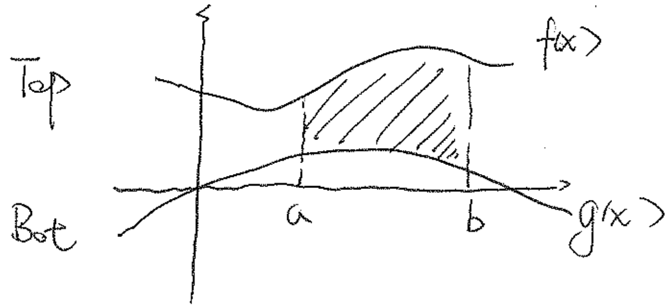
§5.1 Area between curves

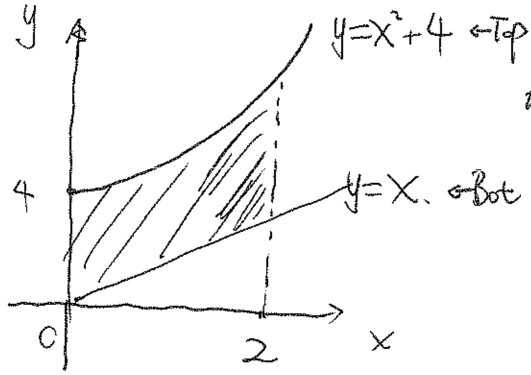
① Two curves: $y=f(x)$, $y=g(x)$ from $x=a$ to $x=b$



Area between the top curve and bot curve is given by

$$\int_a^b [f(x)-g(x)]\,dx.$$

e.g. 1. Sketch the region bounded by $y=x^2+4$, $y=x$, $x=0$, $x=2$ and find the area of the region.



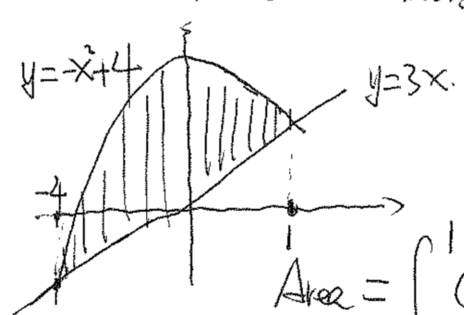
$$\text{Area} = \int_0^2 (x^2+4) - x\, dx.$$

$$= \frac{1}{3}x^3 + 4x - \frac{1}{2}x^2 \Big|_0^2$$

$$= \frac{1}{3}2^3 + 4\cdot 2 - \frac{1}{2}2^2 - (0+0-0) = \boxed{\frac{8}{3}+6}$$

☆ eg. 2. sketch the region bounded by $y=-x^2+4$ and $y=3x$.

Find the intersections of the two curves and find the area of the region



Intersections: $-x^2+4=3x \Leftrightarrow x^2+3x-4=0$

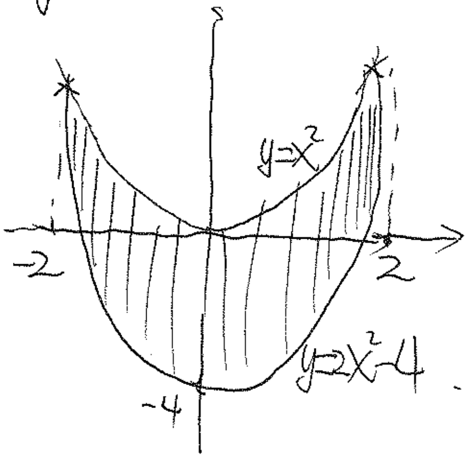
$\boxed{x=-4,\ x=1}$ $\quad (x+4)(x-1)=0$

$$\text{Area} = \int_{-4}^{1} (-x^2+4) - (3x)\,dx.$$

$$= -\frac{1}{3}x^3 + 4x - \frac{3}{2}x^2 \Big|_{-4}^{1} = -\frac{1}{3}+4-\frac{3}{2} - \left(-\frac{1}{3}(-4)^3 + 4(-4) - \frac{3}{2}\cdot(-4)^2\right)$$

$$= \boxed{-\frac{65}{3}+44-\frac{3}{2}}$$

eg 3. Find the area of the region bounded by $y=x^2$, $y=2x^2-4$.



Intersections: $x^2 = 2x^2 - 4$

$\Leftrightarrow \quad x^2 = 4 \quad \Leftrightarrow \quad x = \pm 2$.

Top curve: $y = x^2$.

Bot curve: $y = 2x^2 - 4$

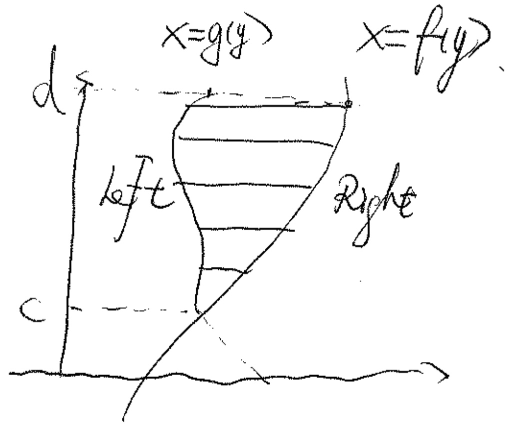
$$\text{Area} = \int_{-2}^{2} x^2 - (2x^2 - 4)\,dx.$$

$$= \int_{-2}^{2} -x^2 + 4\,dx = -\tfrac{1}{3}x^3 + 4x \Big|_{-2}^{2}$$

$$= -\tfrac{1}{3}2^3 + 4\cdot 2 - \left[-\tfrac{1}{3}(-2)^3 + 4(-2)\right]$$

$$= -\tfrac{8}{3} + 8 - \tfrac{8}{3} + 8 = \boxed{16 - \tfrac{16}{3}}$$

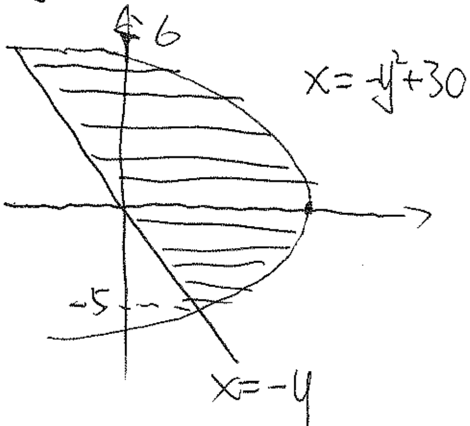
②



$$\text{Area} = \int_c^d \text{Right} - \text{Left}\; dy$$

$$= \int_c^d f(y) - g(y)\,dy \quad (\text{ww}7\text{-}10).$$

eg 4. (ww8): Find the area of the region bounded by $x=-y^2+30$, $x=-y$.



Intersections: $-y^2+30 = -y \Leftrightarrow y^2 - y - 30 = 0$

$y = 6$, $y = -5$.

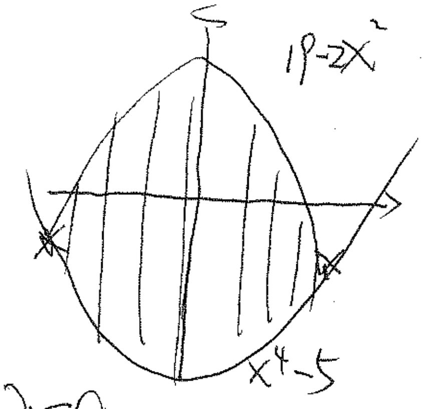
$$\text{Area} = \int_{-5}^{6} (-y^2+30) - (-y)\,dy$$

$$= -\tfrac{1}{3}y^3 + 30y + \tfrac{1}{2}y^2 \Big|_{-5}^{6} = \boxed{-\tfrac{341}{3} + 330 + \tfrac{11}{2}}$$

More hints on Webwork:

eg. 5. (ww 4).

sketch the region bounded by the curves $2x^2+y=19$ and $x^4-y=5$, then find the area of the region.



sln: $2x^2+y=19 \Rightarrow y=19-2x^2 \leftarrow$ Top

$x^4-y=5 \Rightarrow y=x^4-5 \leftarrow$ Bot.

Intersections: $19-2x^2=x^4-5$.

☆ $\Leftrightarrow x^4+2x^2-24=0 \Leftrightarrow (x^2+6)(x^2-4)=0$

$\Leftrightarrow x^2-4=0 \Leftrightarrow x=\pm 2$.

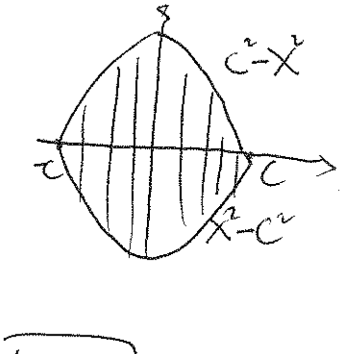
$$\text{Area}=\int_{-2}^{2} 19-2x^2-(x^4-5)\,dx=\int_{-2}^{2} 24-2x^2-x^4\,dx$$

$$=24x-\frac{2}{3}x^3-\frac{1}{5}x^5\Big|_{-2}^{2}=\boxed{\frac{1088}{15}}$$

eg. 6 (ww6).

Find $c>0$ such that the area of the region bounded by $y=x^2-c^2$ and $y=c^2-x^2$ is 19

sln: Intersections: $x^2-c^2=c^2-x^2 \Rightarrow 2x^2=2c^2 \Rightarrow x^2=c^2 \Rightarrow x=\pm c$.



$$\text{Area}=\int_{-c}^{c} (c^2-x^2)-(x^2-c^2)\,dx$$

$$=\int_{-c}^{c} 2c^2-2x^2\,dx$$

$$=2c^2\cdot x-\frac{2}{3}\cdot x^3\Big|_{-c}^{c}$$

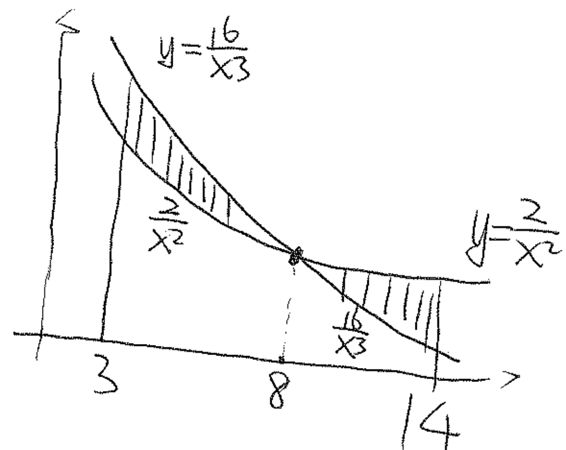
$$=2c^2\cdot c-\frac{2}{3}\cdot c^3-\left[2c^2\cdot(-c)-\frac{2}{3}\cdot(-c)^3\right]=\boxed{\frac{8}{3}c^3}$$

Set $\frac{8}{3}c^3=19 \Rightarrow c^3=\frac{3\cdot 19}{8} \Rightarrow \boxed{c=\left(\frac{3\cdot 19}{8}\right)^{\frac{1}{3}}}$

★★. eg 7 (ww 5)

Find the area of the region bounded by $y=\frac{16}{x^3}$, $y=\frac{2}{x^2}$, $x=3$, $x=14$

sln: Sketch the graph:



The two curves intersect at

$y=\frac{16}{x^3}=\frac{2}{x^2}$

$\Rightarrow 16x^2=2x^3$

$\Rightarrow \frac{16x^2}{2x^2}=\frac{2x^3}{2x^2}$

$\Rightarrow 8=x$

From $x=3$ to $x=8$, $y=\frac{16}{x^3}$ is the top curve
$y=\frac{2}{x^2}$ is the bot curve

From $x=8$ to $x=14$, $y=\frac{16}{x^3}$ is the bot curve
$y=\frac{2}{x^2}$ is the top curve

$$\text{Area}=\int_3^8 \frac{16}{x^3}-\frac{2}{x^2}\,dx+\int_8^{14}\frac{2}{x^2}-\frac{16}{x^3}\,dx.$$

$$=\int_3^8 16x^{-3}-2x^{-2}\,dx+\int_8^{14}2x^{-2}-16x^{-3}\,dx$$

$$=\left(16\cdot\frac{1}{-2}x^{-2}-2\cdot\frac{1}{-1}x^{-1}\right)\Big|_3^8+\left(2\cdot\frac{1}{-1}x^{-1}-16\cdot\frac{1}{-2}x^{-2}\right)\Big|_8^{14}$$

$$=\frac{16}{-2}\cdot 8^{-2}+2\cdot 8^{-1}-\left(\frac{16}{-2}\cdot 3^{-2}+2\cdot 3^{-1}\right)+\left(-2\cdot 14^{-1}+8\cdot 14^{-2}\right)-\left(-2\cdot 8^{-1}+8\cdot 8^{-2}\right)$$

$$=\boxed{-8\cdot\frac{1}{64}+2\cdot\frac{1}{8}+8\cdot\frac{1}{9}-2\cdot\frac{1}{3}+\frac{-2}{14}+\frac{8}{14^2}+2\cdot\frac{1}{8}-8\cdot\frac{1}{64}}$$